Finding a hole in the plane with eyes closed!

Institute Colloquium, IISER Tirupati

Gadadhar Misra

Indian Statistical Institute Bangalore and IIT Gandhinagar

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The Fundamental Theorem of Calculus

Recall the two forms of the fundamental theorem of calculus.

- The first form says that continuous $f : [a,b] \to \mathbb{R}$ admits a primitive: there exists $F : (a,b) \to \mathbb{R}$ such that $F' = f$ on (a,b).
- The second form says: if g is continuous on $[a,b]$, differentiable on (a,b), and g' is continuous on $[a,b]$, then

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Extension to $\mathbb{R}^n$

What happens if the functions are defined on an open subset U of $\mathbb{R}^n$?

If $f : U \to \mathbb{R}^2$ is continuous, does there exist a continuously differentiable $F : U \to \mathbb{R}$ such that

$$\frac{\partial F}{\partial x_1} = f_1, \quad \frac{\partial F}{\partial x_2} = f_2? \qquad (\star)$$

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A necessary condition

If such F exists, then we must have

$$\frac{\partial^2 F}{\partial x_2 \partial x_1} = \frac{\partial f_1}{\partial x_2}, \quad \frac{\partial^2 F}{\partial x_1 \partial x_2} = \frac{\partial f_2}{\partial x_1}. \qquad (\dagger)$$

- Therefore, a necessary condition is:

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$$\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1}. \tag{‡}$$

Vector fields and potentials

The function f is called a vector field. If F satisfies $(\dagger)$, it is the potential (or primitive) of f, and f is said to be conservative.

A necessary condition for conservativeness is $(\ddagger)$.

Question. Is the necessary condition also sufficient?

We will show that, in general, the answer is negative. Under certain topological conditions on U, the necessary condition becomes sufficient.

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The counterexample

Let $U = \mathbb{R}^2 \setminus \{(0,0)\}$ and define $f : U \to \mathbb{R}^2$:

$$f(x_1, x_2) = \left(\frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right).$$

Checking the necessary condition

Check that (‡) is satisfied:

$$\frac{\partial f_2}{\partial x_1} = \frac{x_1^2 + x_2^2 - 2x_1^2}{(x_1^2 + x_2^2)^2} = \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2},$$

$$\frac{\partial f_1}{\partial x_2} = \frac{-x_1^2 - x_2^2 + 2x_2^2}{(x_1^2 + x_2^2)^2} = \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}.$$

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f is not conservative

Suppose there exists a continuously differentiable $F : U \to \mathbb{R}$ with $\frac{\partial F}{\partial x_1} = f_1$, $\frac{\partial F}{\partial x_2} = f_2$.

- Then

$$\int_0^{2\pi} \frac{d}{d\theta} F(\cos\theta, \sin\theta)\, d\theta = F(\cos\theta, \sin\theta)\Big|_0^{2\pi}$$

$$= F(1,0) - F(1,0) = 0.$$

We evaluate the same integral using a different method and arrive at a contradiction.

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The Chain Rule

Lemma (Chain Rule)

Suppose that $\Theta : \mathbb{R} \to \mathbb{R}^2$ and $F : \mathbb{R}^2 \to \mathbb{R}$ are differentiable functions. Then

$$D\big(F \circ \Theta\big)(t) = DF(\Theta(t))\, D\Theta(t), \quad t \in \mathbb{R}. \qquad (\natural)$$

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Setting up the computation

Let $\Theta(t) = (\cos t, \sin t)$. For any $(a,b) \in \mathbb{R}^2$:

$$DF(a,b)\begin{bmatrix} u \\ v \end{bmatrix} = u\frac{\partial F}{\partial x_1}(a,b) + v\frac{\partial F}{\partial x_2}(a,b).$$

- For any $t \in \mathbb{R}$:

$$D\Theta(t)\,r = \begin{bmatrix} -r\sin t \\ r\cos t \end{bmatrix}, \quad r \in \mathbb{R}.$$

- Note: (i) $f_1(\cos\theta, \sin\theta) = -\sin\theta$ and (ii) $f_2(\cos\theta, \sin\theta) = \cos\theta$.

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Applying the chain rule

$$\begin{aligned}
D(F \circ \Theta)(t) &= DF(\Theta(t))\, D\Theta(t) \\
&= DF(\cos t, \sin t) \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \\
&= -\sin t \frac{\partial F}{\partial x_1}(\cos t, \sin t) + \cos t \frac{\partial F}{\partial x_2}(\cos t, \sin t) \\
&= -\sin t f_1(\cos t, \sin t) + \cos t f_2(\cos t, \sin t) \\
&= \sin^2 t + \cos^2 t = 1.
\end{aligned}$$

The contradiction

Since $D(F \circ \Theta)(t) = \frac{d}{dt}F(\cos t, \sin t)$, it follows that

$$\int_0^{2\pi} \frac{d}{dt}F(\cos t, \sin t)\, dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

But we showed that this integral equals 0. Contradiction!

Therefore, although f satisfies $(\ddagger)$, there is no F satisfying $(\star)$. The vector field f is not conservative.

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Star-shaped domains

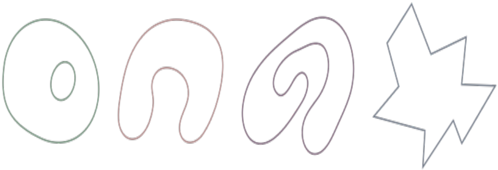
When is the necessary condition for the existence of a potential also sufficient?

Definition

A domain $U \subset \mathbb{R}^2$ is star shaped if there is a point $x_0 \in U$ such that

$$\{tx_0 + (1-t)x \mid t \in [0,1]\} \subset U$$

for all $x \in U$.



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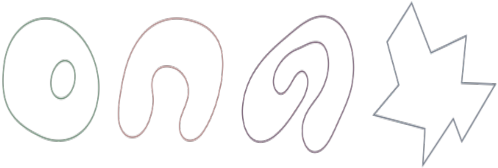

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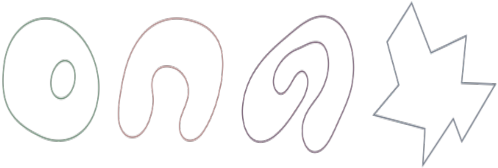

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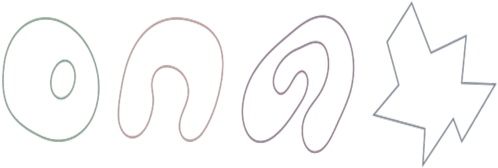

The Poincaré Lemma

Theorem

Let $U \subset \mathbb{R}^2$ be a star shaped open bounded set. Any continuously twice differentiable function $f : U \to \mathbb{R}^2$ satisfying $(\ddagger)$ has a potential F.

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Let $U \subset \mathbb{R}^2$ be a star shaped open bounded set. Any continuously twice differentiable function $f : U \to \mathbb{R}^2$ satisfying $(\ddagger)$ has a potential F.

Proof: the formula for F

WLOG, $(0,0) \in U$ and U is star shaped relative to $(0,0)$.

Define $F : U \to \mathbb{R}$ by:

$$F(x_1,x_2) = \int_0^1 \big(x_1 f_1(tx_1,tx_2) + x_2 f_2(tx_1,tx_2)\big)\,dt.$$

Claim: $\dfrac{\partial F}{\partial x_1}(x_1,x_2) = f_1(x_1,x_2).$

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Proof: differentiating $f(tx_1, tx_2)$ w.r.t. x_1

Think of $f(tx_1, tx_2)$ as a composition:

$$(x_1, x_2) \xrightarrow{T} (tx_1, tx_2) \xrightarrow{h} f(tx_1, tx_2).$$

The derivative of the linear map T is $DT(x_1, x_2) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$.
By the chain rule applied to $h \circ T$:

$$\begin{aligned} D\big(h \circ T\big)(x_1, x_2) &= Dh\big(T(x_1, x_2)\big) DT(x_1, x_2) \\ &= \left(\frac{\partial h}{\partial x_1}(tx_1, tx_2), \frac{\partial h}{\partial x_2}(tx_1, tx_2)\right) \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \\ &= \left(t\frac{\partial h}{\partial x_1}(tx_1, tx_2), t\frac{\partial h}{\partial x_2}(tx_1, tx_2)\right). \end{aligned}$$

Thus, applying this formula to f_1, we have
$\frac{\partial f_1}{\partial x_1}(tx_1, tx_2) = t\frac{\partial f_1}{\partial x_1}(tx_1, tx_2)$, similarly, for f_2.

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Proof: computing $\frac{d}{dt}f_1(tx_1, tx_2)$

Think of $f_1(tx_1, tx_2)$ as:

$$t \xrightarrow{\tau} (tx_1, tx_2) \xrightarrow{f_1} f_1(tx_1, tx_2).$$

We have $D\tau(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. By the chain rule:

$$\frac{d}{dt}f_1(tx_1, tx_2) = x_1 \frac{\partial f_1}{\partial x_1}(tx_1, tx_2) + x_2 \frac{\partial f_1}{\partial x_2}(tx_1, tx_2).$$

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Proof: the product rule

Applying the product rule:

$$\frac{d}{dt}\big(tf_1(tx_1,tx_2)\big) = f_1(tx_1,tx_2) \\ + tx_1\frac{\partial f_1}{\partial x_1}(tx_1,tx_2) + tx_2\frac{\partial f_1}{\partial x_2}(tx_1,tx_2).$$

Proof: the final computation

Differentiating F under the integral sign w.r.t. x_1:

$$\begin{aligned}\frac{\partial F}{\partial x_1} &= \int_0^1 \left\{ f_1 + t x_1 \frac{\partial f_1}{\partial x_1} + t x_2 \frac{\partial f_2}{\partial x_1} \right\} dt \\ &= \int_0^1 \left\{ \frac{d}{dt}(t f_1) + t x_2 \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) \right\} dt.\end{aligned}$$

(all functions evaluated at (tx_1, tx_2))

By hypothesis, $\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1}$. Therefore:

$$\frac{\partial F}{\partial x_1}(x_1, x_2) = \int_0^1 \frac{d}{dt}\big(t f_1(tx_1, tx_2)\big)\, dt = t f_1(tx_1, tx_2)\Big|_0^1 = f_1(x_1, x_2).$$

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The other half of the FTC

What about the second form of the fundamental theorem?

We must first discuss integration of functions on open sets and curves in $\mathbb{R}^2$.

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Double integrals

Let $U \subset [a,b] \times [c,d]$ be open and connected. Let $\triangle x_i = x_{i+1} - x_i$, $\triangle y_j = y_{j+1} - y_j$ be determined by partitions of $[a,b]$ and $[c,d]$.

For continuous f on U, the integral of f over U is:

$$\iint_U f\,dx\,dy := \lim_{\triangle x_i, \triangle y_j \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j) \triangle x_i \triangle y_j,$$

if the limit exists.

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Path integrals

Let $F : U \to \mathbb{R}^2$ be continuously differentiable and $\gamma : [0,1] \to \mathbb{R}^2$ continuously differentiable with $\gamma'(t) > 0$ and $\partial U = \operatorname{ran}\gamma$.

The path integral of F over γ is:

$$\int_{\partial U} F = \int_0^1 F(t) \cdot \gamma'(t)\, dt.$$

A region U is called simple if ∂U is parametrized by a continuously differentiable $\gamma : [a,b] \to U$ tracing ∂U once counterclockwise.

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Green's Theorem

If U is simple and $F := (F_1, F_2) : U \cup \partial U \to \mathbb{R}^2$ is continuously differentiable, then:

$$\iint_U \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dx_1\, dx_2 = \int_{\partial U} F.$$

This is the fundamental theorem of calculus for $\mathbb{R}^2$: the integrand on the left is a "derivative" and $\int_{\partial U} F$ generalizes $\int_a^b f$ to curves in $\mathbb{R}^2$.

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Detecting holes: grad and curl

The final task: find invariants for U to detect if it has a hole.

Let $C^\infty(U,\mathbb{R}^k)$ be the space of smooth functions. Define:

$$\text{grad}: C^\infty(U,\mathbb{R}) \to C^\infty(U,\mathbb{R}^2), \quad \text{grad}(F) = \begin{bmatrix} \frac{\partial F}{\partial x_1} \\ \frac{\partial F}{\partial x_2} \end{bmatrix},$$

$$\text{curl}: C^\infty(U,\mathbb{R}^2) \to C^\infty(U,\mathbb{R}), \quad \text{curl}(f) = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}.$$

- It is easily verified that $\text{curl} \circ \text{grad} = 0$.

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The de Rham complex

Since $\mathrm{curl} \circ \mathrm{grad} = 0$, we have $\ker(\mathrm{curl}) \supseteq \mathrm{Im}(\mathrm{grad})$. The diagram

$$\{0\} \longrightarrow C^\infty(U,\mathbb{R}) \stackrel{\mathrm{grad}}{\longrightarrow} C^\infty(U,\mathbb{R}^2) \stackrel{\mathrm{curl}}{\longrightarrow} C^\infty(U,\mathbb{R}) \longrightarrow \{0\}$$

is a complex. The quotient

$$H^1(U) := \ker(\mathrm{curl}) \big/ \mathrm{Im}(\mathrm{grad})$$

is, quite surprisingly, finite dimensional.

- If U is star shaped, then $H^1(U) = \{0\}$.
- If $U = \mathbb{R}^2 \setminus \{(0,0)\}$, then $H^1(U) \neq \{0\}$.

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Thank You!